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MATHEMATICAL MODELING OF FLOW IN CONFINED AQUIFER

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Abstract: **The general equation is developed for the simple one-dimensional case, and then the results are extended to the three-dimensional case. We have derived equation in Cartesian coordinate system and radial flow in an aquifer for confined aquifer and also given a simple analytical solution for estimation of drawdowns and groundwater flow rates into two-dimensional excavation, such as those in open-cut strip mines, for confined, leaky aquifers.**

Keywords: **Aquifer, Confined, Homogeneous, Saturated, Transmissivity, Two-dimensional.**

1. INTRODUCTION

The first step in developing a mathematical model of almost any system is to formulate what are known as general equations. General equations are differential equations that derive from the physical principles governing the process that is to be modeled. In the case of subsurface flow, the relevant physical principles are Darcy's law and mass balance. By combining the mathematical relations describing these principles, it is possible to come up with a general groundwater flow equation, which is partial differential equation. If a mathematical model obeys the general equation, it is consistent with Darcy's law and mass balance. For anyone using or developing models, it is helpful to understand the general equation and how it relates to the underlying physical principles. There are several different forms of the general flow equation depending on whether the flow is saturated or unsaturated, Two-dimensional or three-dimensional, isotropic or anisotropic, and transient or steady state.

2. MODELING OF THREE-DIMENSIONALSATURATED FLOW

First, the general equation is developed for the simple one-dimensional case, and then the results are extended to the three-dimensional case [1]. In a typical mass balance analysis, the net flux of mass through the boundary of an element is equated to the rate of change of mass within the element. We will consider the mass balance for a small rectangular element within the saturated zone. The dimensions of the element are fixed in space, regardless of compression or dilation of the aquifer matrix. For example, if the aquifer compresses, more aquifer solids will be squeezed into the element and some water will be squeezed out of it. To make the derivation of the flow equations as clear as possible, we will assume that the macroscopic flow in the vicinity of this element is one-dimensional in the x direction: $q_x \neq 0, q_y = q_z = 0$. The mass flux (mass/time) of water in through the left side of the element is

$$\rho_w(x) q_x(x) \Delta y \Delta z \qquad (2.1)$$

Where $\rho_w(x)$ the water density at is coordinate x and $q_x(x)$ is the specific discharge at coordinate x. The corresponding flux out through the right side of the element is

$$\rho_w(x + \Delta x)\, q_x(x + \Delta x) \Delta y \Delta z \qquad (2.2)$$

When these two fluxes are identical, the flow is steady state. When they differ, the flow is transient and there must be a change in the mass of water stored in the element. According to the definition of specific storage S_s, the change in the volume of water stored in an element of volume V_t when the head changes an amount dh is

$$dV_w = S_s dh V_t \qquad (2.3)$$

For the element and a time interval ∂t, this becomes

$$\frac{\partial V_w}{\partial t} = S_s \frac{\partial h}{\partial t} \Delta x \Delta y \Delta z \qquad (2.4)$$

The rate of change in the mass of water stored in the element is therefore

$$\frac{\partial m}{\partial t} = \rho_w \frac{\partial V_w}{\partial t} = \rho_w S_s \frac{\partial h}{\partial t} \Delta x \Delta y \Delta z \qquad (2.5)$$

The rate $\partial h/\partial t$ is expressed as a partial derivative because in this case, h is a function of two variables, x and t. Equations (2.1), (2.2), and (2.5) describe all the mass fluxes [M/T] into the element. For mass balance or continuity, the mass flux into the element minus the mass flux out equals the rate of change of mass stored within the element.

$$\rho_w(x) q_x(x) \Delta y \Delta z - \rho_w(x+\Delta x) q_x(x+\Delta x) \Delta y \Delta z = \rho_w S_s \frac{\partial h}{\partial t} \Delta x \Delta y \Delta z \qquad (2.6)$$

Dividing by $\Delta x \Delta y \Delta z$ and rearranging gives

$$-\left[\frac{\rho_w(x+\Delta x) q_x(x+\Delta x) - \rho_w(x) q_x(x)}{\Delta x}\right] = \rho_w S_s \frac{\partial h}{\partial t} \qquad (2.7)$$

Recalling some differential calculus, the left-hand side is a derivative in the limit as Δx shrinks to zero.

$$-\frac{\partial(\rho_w q_x)}{\partial x} = \rho_w S_s \frac{\partial h}{\partial t} \qquad (2.8)$$

Expanding the derivative on the left side of the above equation, it becomes

$$-\rho_w \frac{\partial q_x}{\partial x} - q_x \frac{\partial \rho_w}{\partial x} = \rho_w S_s \frac{\partial h}{\partial t} \qquad (2.9)$$

The second term in the above equation is generally orders of magnitude smaller than the first one.

$$\rho_w \frac{\partial q_x}{\partial x} \gg q_x \frac{\partial \rho_w}{\partial x} \qquad (2.10)$$

Neglecting the second term in Eq. (2.9), the continuity condition can be simplified to

$$-\frac{\partial q_x}{\partial x} = S_s \frac{\partial h}{\partial t} \qquad (2.11)$$

In most situations, Equation (2.10) is true and the above equation governs, but more rigorous theories may be needed for flow in special circumstances. Derivations of more rigorous general flow equations are given by Freeze and Cherry (1979), Verruijt (1969), and Gambolati (1973, 1974); these account for the velocity of the deforming matrix in very low conductivity materials and fluid density variations. Substituting the definition of q_x given by Darcy's law that is $q_x = -K_x \frac{\partial h}{\partial x}$, into Equation (2.11) gives the one-dimensional general equation for saturated groundwater flow[3].

$$\frac{\partial}{\partial x}\left(K_x \frac{\partial h}{\partial x}\right) = S_s \frac{\partial h}{\partial t} \qquad (2.12)$$

If the preceding analysis were carried out without the restriction of one-dimensional flow, there would be additional flux terms for the y and z directions that are similar to the flux term for the x direction. For three-dimensional flow, the general equation is

$$\frac{\partial}{\partial x}\left(K_x \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial h}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_z \frac{\partial h}{\partial z}\right) = S_s \frac{\partial h}{\partial t} \qquad (2.13)$$

A mathematical model of head $(h(x, y, z, t) = \ \ldots)$ must obey this partial differential equation if it is to be consistent with Darcy's law and mass balance. Equation (2.13) is the most universal form of the saturated flow equation, allowing flow in all three directions, transient flow $(\partial h/\partial t \neq 0)$, heterogeneous conductivities (for example, $K_x = f(x)$), and anisotropic hydraulic conductivity $(K_x \neq K_y \neq K_z)$.

3. HOMOGENEOUS HYDRAULIC CONDUCTIVITIES

The flow equation can be derived from Eq. (2.13) by making various simplifying assumptions. If the hydraulic conductivity is homogeneous (i.e. independent of $x, y, and\ z$), then general equation can be written as [1].

$$K_x \frac{\partial^2 h}{\partial x^2} + K_y \frac{\partial^2 h}{\partial y^2} + K_z \frac{\partial^2 h}{\partial z^2} = S_s \frac{\partial h}{\partial t} \qquad (3.1)$$

4. HOMOGENEOUS AND ISOTROPIC HYDRAULIC CONDUCTIVITIES

When K is assumed to be both homogeneous and isotropic ($K_x = K_y = K_z = K$), the Eq. (3.1) reduces to

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} = \nabla^2 h = \frac{S_s}{K} \frac{\partial h}{\partial t} \qquad (4.1)$$

The symbol ∇^2 is called the Laplacian operator, and it is shorthand for the sum of the second derivatives,

$$\nabla^2(\,) = \frac{\partial^2(\,)}{\partial x^2} + \frac{\partial^2(\,)}{\partial y^2} + \frac{\partial^2(\,)}{\partial z^2} \qquad (4.2)$$

5. STEADY FLOW WITH HOMOGENEOUS, ISOTROPIC HYDRAULIC CONDUCTIVITIES

So far, all forms of the general flow equation presented here have included a term for storage changes involved with transient flow. If instead the flow is steady state, $\partial h/\partial t$= 0 and the right-hand side of any of the previous equations becomes zero. For example, the general equation for steady flow with homogeneous, isotropic K is [3].

$$\nabla^2 h = 0 \qquad (5.1)$$

This is a common partial differential equation known as the Laplace equation. It is well studied, having numerous applications in fluid flow, heat conduction, electrostatics, and elasticity. It is named after French astronomer and mathematician Pierre de Laplace (1749-1827). There exist hundreds of known solutions to the Laplace equation, many of which apply directly to common groundwater flow conditions**.** Any of the flow equations presented here can be reduced from three dimensions to two or one by dropping the y and /or z terms from the equation. Dropping the z dimension, for example, implies that the z-direction term in the general equation equals zero.

$$\frac{\partial}{\partial z}\left(K_z \frac{\partial h}{\partial z}\right) = 0 \qquad (5.2)$$

This would be the case if $q_z = 0$, or even if $\partial h/\partial z = 0$.

6. CONFINED OR UNCONFINED AQUIFER

The flow towards a well, situated in homogeneous and isotropic confined or unconfined aquifer is radially symmetric. The cone of depression caused due to constant pumping through a single well situated at (0,0) in a confined aquifer. The cone of impression caused due to constant recharge through the well. In case of homogeneous and isotropic medium, the cone of depression or cone of impression is radially symmetrical. The governing equation derived earlier in Cartesian coordinate system for confined and unconfined aquifer can also be derived for radial flow in an aquifer. We will derive the governing flow equation for confined aquifer in polar coordinate system. The main objective of this conversion is to make the 2D flow problem a 1D flow problem. The resulting 1D problem will be simpler to solve.

7. RADIAL FLOW IN A CONFINED AQUIFER

Let us consider a case of radial flow to a single well in a confined aquifer. The aquifers are homogeneous and isotropic and have constant thickness of b. The hydraulic conductivity of the aquifer is K. The pumping rate (Q) of the aquifer is constant and the well diameter is infinitesimally small. The well is fully penetrated into the entire thickness of the confined aquifer. This is necessary to make the flow essentially horizontal. The potential head in the aquifer prior to pumping is uniform throughout the aquifer [7]. Consider the control volume. The inflow to the control volume is Q_r and the outflow from the control volume is $Q_r + \frac{\partial Q_r}{\partial r} dr$. The net inflow to the control volume is

$$Q_r - \left(Q_r + \frac{\partial Q_r}{\partial r} dr\right) = -\frac{\partial Q_r}{\partial r} dr \qquad (7.1)$$

Applying principle of mass conservation on the control volume.

Inflow - outflow = Time rate of change in volumetric storage

Time rate of change in volumetric storage $= \frac{\partial V}{\partial t} = V\left(\frac{\partial V}{V\partial h}\right)\frac{\partial h}{\partial t}$ (7.2)

$$\frac{\partial V}{\partial t} = VS_0 \frac{\partial h}{\partial t} \qquad (7.3)$$

$$= \frac{\partial V}{V\partial h}$$

Where S_0 is the specific storage. Replacing V by $2\pi rdrb$, we have

$$\frac{\partial V}{\partial t} = VS_0 \frac{\partial h}{\partial t} = 2\pi rdrbS_0 \frac{\partial h}{\partial t} \qquad (7.4)$$

$$= 2\pi rdrb \frac{S_s}{b}\frac{\partial h}{\partial t} = 2\pi rdrS_s \frac{\partial h}{\partial t} \qquad (7.5)$$

Where S_s is the aquifer storativity which is equal to S_0/b. Putting (7.5) in (7.3), we have

$$-\frac{\partial Q_r}{\partial r} dr = 2\pi rdrS_s \frac{\partial h}{\partial t} \qquad (7.6)$$

As per Darcy's law

$$Q_r = -KA\frac{\partial h}{\partial r} = -K2\pi rb\frac{\partial h}{\partial r} = -2\pi rT\frac{\partial h}{\partial r} \qquad [\text{Putting } T = Kb\,] \qquad (7.7)$$

Putting in equation (7.6)

$$\frac{\partial}{\partial r}\left(2\pi rT\frac{\partial h}{\partial r}\right) dr = 2\pi rdrS_s \frac{\partial h}{\partial t} \qquad (7.8)$$

Simplifying,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial h}{\partial r}\right) = \frac{S_s}{T}\frac{\partial h}{\partial t} \qquad (7.9)$$

$$\frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} = \frac{S_s}{T}\frac{\partial h}{\partial t} \qquad (7.10)$$

This is the flow equation for radial flow into a well for confined homogeneous and isotropic aquifer. In case of steady state condition, the governing equation becomes,

$$\frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} = 0 \qquad (7.11)$$

8. STEADY FLOW IN CONFINED AQUIFER

In case of steady flow in confined aquifer, the flow equation becomes [5].

Or, $$\frac{1}{r}\frac{d}{dr}\left(r\frac{dh}{dr}\right) = 0 \qquad (8.1)$$

Or, $$\frac{d}{dr}\left(r\frac{dh}{dr}\right) = 0 \qquad (8.2)$$

Integrating, $$\int \frac{d}{dr}\left(r\frac{dh}{dr}\right) = \int 0 \qquad (8.3)$$

Or, $$r\frac{dh}{dr} = C_1 \qquad (8.4)$$

Now, Darcy's law can be expressed as

$$Q = 2\pi rT\frac{dh}{dr} \rightarrow \frac{Q}{2\pi T} = r\frac{dh}{dr} = C_1 \qquad (8.5)$$

Therefore, the equation (11.4) can be written as

$$r\frac{dh}{dr} = \frac{Q}{2\pi T} \qquad (8.6)$$

$$\Rightarrow dh = \frac{Q}{2\pi T}\frac{dr}{r} \qquad (8.7)$$

Now integrating, we have

$$\Rightarrow \int dh = \frac{Q}{2\pi T}\int \frac{dr}{r} \qquad (8.8)$$

$$\Rightarrow h = \frac{Q}{2\pi T}\ln(r) + C_2 \qquad (8.9)$$

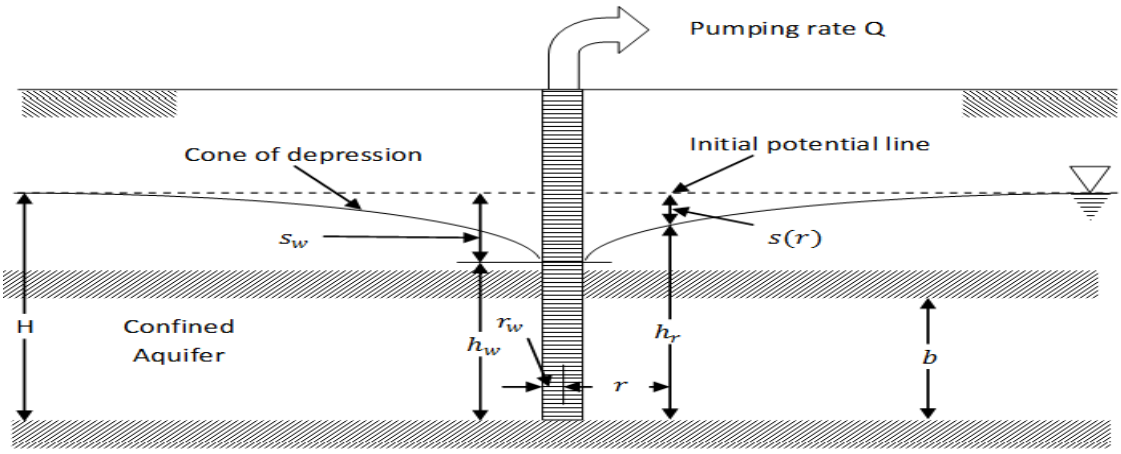

Fig. 8.1: Confined aquifer

Now consider the Fig. 8.1. For $r = r_w \rightarrow h = h_w \; and \; r = r \rightarrow h = h_r$ Putting it in equation (8.4), we get

$$h_w = \frac{Q}{2\pi T}\ln(r_w) + C_2 \qquad (8.10)$$

and, $$h_r = \frac{Q}{2\pi T}\ln(r) + C_2 \qquad (8.11)$$

From these two equations, we have

$$h_r - h_w = \frac{Q}{2\pi T}\ln(r) - \frac{Q}{2\pi T}\ln(r_w) \qquad (8.12)$$

$$\Rightarrow h_r - h_w = \frac{Q}{2\pi T}\ln\left(\frac{r}{r_w}\right) \qquad (8.13)$$

Knowing hydraulic head at the well, the equation (8.13) can be used to calculate steady state hydraulic head for any values of r. This equation can also be used for estimation of aquifer transmissivity. For calculating aquifer transmissivity, the equation can be written as,

$$T = \frac{Q}{2\pi(h_r - h_w)}\ln\left(\frac{r}{r_w}\right) \qquad (8.14)$$

9. FLOW EQUATION FOR UNSTEADY FLOW IN CONFINED AQUIFER

We have already derived the flow equation for unsteady flow in confined aquifer. The equation can be written as [3],

$$\frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} = \frac{S_s}{T}\frac{\partial h}{\partial t} \qquad (9.1)$$

Theis (1935) obtained the solution of the equation. His solution was based on the analogy between groundwater flow and heat conduction. Considering the following boundary conditions,

$at \;\; t = 0 \;\;, \qquad h = h_0$
$at \;\; t = \infty \;\;, \qquad h = h_0$

The solution of the equation for $t \geq 0$ is

$$s(r,t) = \frac{Q}{4\pi T}W(u) \qquad (9.2)$$

Where, $s(r,t)$ is the draw down at a radial distance r from, the well at time t,

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$$u = \frac{r^2 S_s}{4Tt} \quad and \quad W(u) = \int_u^\infty \frac{e^{-u}}{u} du$$

$W(u)$ is the exponential integration and is known as well function. The well function $W(u)$ can be approximated as

$$W(u) = -0.5772 - \ln(u) + u - \frac{u^2}{2.2!} + \frac{u^3}{3.3!} + \frac{u^4}{4.4!} + \cdots \quad (9.3)$$

10. THEIS ANALYTICAL SOLUTION

Theis analytical solution was based on the analogy between groundwater flow and heat conduction. In case of heat conduction, the change in temperature (v) at a point $p(x, y)$ at any time t due to an instantaneous line source(x)coinciding with the Z axis can be obtained using the following equation given by [11] and Carslaw (1921).

$$v(x, y, t) = \frac{x}{4\pi kt} e^{-(x^2+y^2)/4kt} \quad (10.1)$$

Here, k is the Kelvin's coefficient of diffusivity. For continuous source or sink $x(\tau)$

$$v(x, y, t) = \int_0^t \frac{x(\tau)}{4\pi k(t-t')} e^{-(x^2+y^2)/4k(t-\tau)} d\tau \quad (10.2)$$

For constant source $x(\tau) = x$

$$v(x, y, t) = \frac{x}{4\pi k} \int_0^t \left[\frac{e^{-(x^2+y^2)/4k(t-\tau)}}{(t-\tau)}\right] d\tau \quad (10.3)$$

Considering

$$u = \frac{x^2+y^2}{4k(t-\tau)} \quad (10.4)$$

When,

$$\begin{aligned} \tau &= 0, & u &= \frac{x^2+y^2}{4kt} \\ \tau &= t & u &= \infty \\ and & & d\tau &= \frac{x^2+y^2}{4k} \frac{1}{u^2} du \end{aligned} \quad (10.5)$$

Then,

$$v(x, y, t) = \frac{x}{4\pi k} \int_{\frac{x^2+y^2}{4kt}}^\infty \left[\frac{e^{-u}}{(t-\tau)}\right] \frac{x^2+y^2}{4k} \frac{1}{u^2} du \quad (10.6)$$

$$v(x, y, t) = \frac{x}{4\pi k} \int_{\frac{x^2+y^2}{4kt}}^\infty \left[\frac{e^{-u}}{u^2}\right] u du \quad (10.7)$$

$$v(x, y, t) = \frac{x}{4\pi k} \int_{\frac{x^2+y^2}{4kt}}^\infty \frac{e^{-u}}{u} du \quad (10.8)$$

The equation (10.8) derived for calculation of change in temperature can also be applied for calculation of drawdown at any point (x, y) at any time t. The coefficient of diffusivity is analogous to the coefficient of transmissivity of the aquifer divided by the specific storage (S_s) of the aquifer. The continuous strength of the source and sink is analogous to the discharge rate divided by the specific storage. The equation (10.8) in case of drawdown in confined aquifer can be written as

$$s(x, y, t) = \frac{Q/S_s}{4\pi T/S_s} \int_{\frac{x^2+y^2}{4(T/S)t}}^\infty \frac{e^{-u}}{u} du \quad (10.9)$$

$$s(x, y, t) = \frac{Q}{4\pi T} \int_{\frac{S_s(x^2+y^2)}{4Tt}}^\infty \frac{e^{-u}}{u} du \quad (10.10)$$

Putting $x^2 + y^2 = r^2$

$$s(r,t) = \frac{Q}{4\pi T}\int_{\frac{S_s r^2}{4Tt}}^{\infty} \frac{e^{-u}}{u} du \qquad (10.11)$$

Equation (10.11) can be used to calculate the drawdown at a distance of *r* at any time *t* when water is pumped at a constant rate of *Q* from the well. This solution is valid homogeneous isotropic aquifer having infinite areal extent and uniform thickness.

11. WELLS IN A LEAKY CONFINED AQUIFER

A confined aquifer will be called a leaky aquifer when water is withdrawn from the confined aquifer, there is a vertical flow from the overlaying aquitard as shown in Fig. 11.1. After the starts of the pumping, the lowering of piezometric head in the aquifer builds hydraulic gradient within the aquitard. As a result of the hydraulic gradient, downward vertical groundwater flow takes place through the aquitard [9].

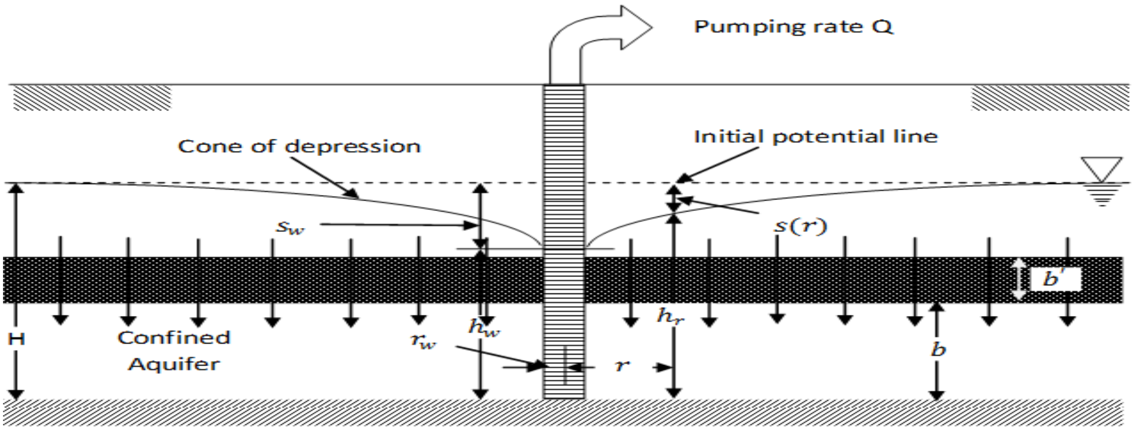

Fig. 11.1 A leaky confined aquifer

The drawdown of the piezometric surface can be obtained by (Hantush 1956, Cobb *et al.* 1982)

$$s = \frac{Q}{4\pi T} W\left(u, \frac{r}{B}\right) \qquad (11.1)$$

Where

$$W\left(u, \frac{r}{B}\right) = \int_0^{\alpha} \exp\{-u - r^2/[4B^2(x+u)]\}/(x+u)\exp(-x)dx \qquad (11.2)$$

$$u = \frac{S_s r^2}{4Tt} \qquad (11.3)$$

$$\frac{r}{B} = \frac{r}{\sqrt{T/(K' b')}} \qquad (11.4)$$

Where *T* is the transmissivity of the leaky confined aquifer, *K'* is the vertical hydraulic conductivity of the aquitard, and *b'* is the thickness of the aquitard.

12. PARTIALLY PENETRATING WELL

In a well when the intake of the well is less than the thickness of the well, then the well is called partially penetrated well. In case of partially penetrated well, the flow lines are not truly horizontal near the well. The flow lines are curved upward or downward near the well. However, at a distance far away from the well, the flow lines are horizontal. As a result of non-horizontal nature of the flow lines near the well, the length of the flow lines are more than the case of a fully penetrated well. Thus the drawdown in case of partially penetrating well is more than the fully penetrating well. Fig. 12.1 shows a partially penetrated well [3].

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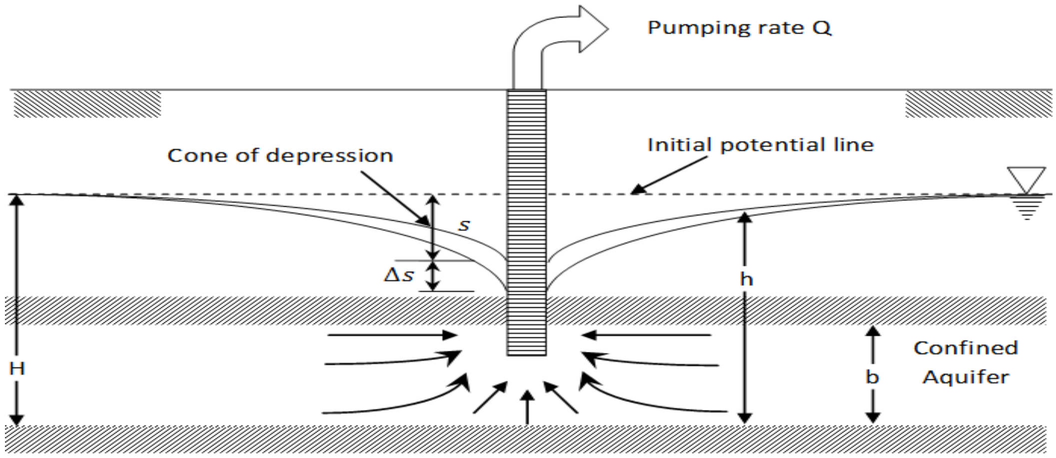

Fig. 12.1 Partially penetrated well

The drawdown of the partially penetrated well can be written as

$$s_p = s + \Delta s \qquad (12.1)$$

Where, S is the drawdown of the fully penetrated well and Δ_s is the additional drawdown due to partial penetration. For the Fig. 12.2 given below,

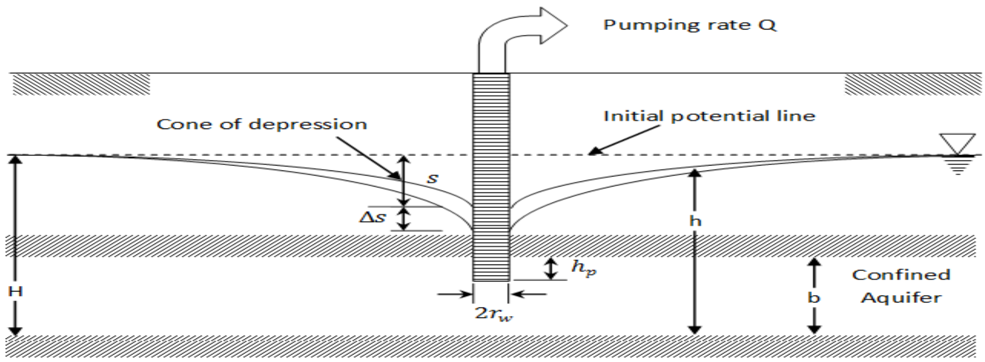

Fig. 12.2 Partially penetrated well

The additional drawdown, Δ_s can be calculated as [12]

$$\Delta s = \frac{Q}{2\pi T}\frac{1-p}{p}\ln\left(\frac{(1-p)h_p}{r_w}\right) \qquad (12.2)$$

13. CHANGE IN HYDRAULIC PROPERTIES NEAR A WELL

Consider a case of a pumping well as shown in Fig. 13.1 below [9].

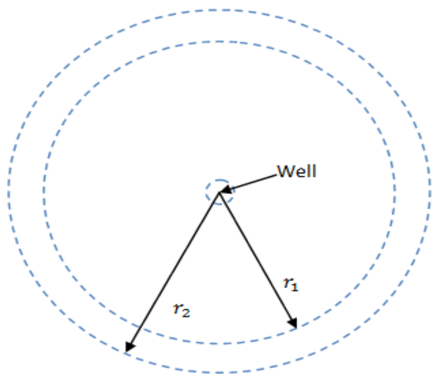

Fig. 13.1 Pumping well

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The discharge of the well can be expressed as

$$Q = A_1V_1 = A_2V_2 \qquad (13.1)$$

$$\Rightarrow 2\pi r_1 hV_1 = 2\pi r_2 hV_2 \qquad (13.2)$$

$$\Rightarrow r_1V_1 = r_2V_2 \qquad (13.3)$$

Here $r_2 > r_1$

So $V_1 > V_2$

Therefore, velocity near the well is more than the velocity away from the well. Due to the high velocity in the vicinity of the well , the fine particles that are present in the aquifer formation are moved with the flow of water. As a result of this phenomenon, the permeability of the aquifer medium will be more in the vicinity of the well.

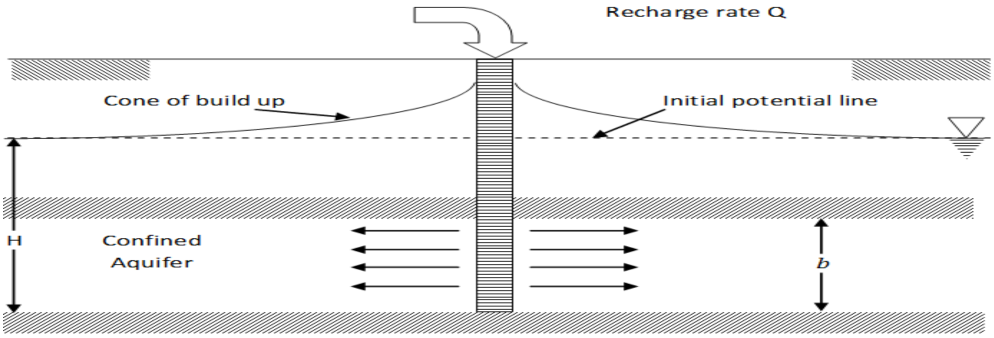

Fig. 13.2 Recharge well

Now, in case of recharge well (Fig. 13.2), the impurities that are present in water are also move along with the injected water to the aquifer medium. As the velocity of flow in the vicinity of the well is higher, the impurities present in the water will move along with the water and will settle down at some distance from the well. As a result of the settlement of impurities, the permeability of the medium will reduce. As such the reduction on permeability should be considered in modeling the flow in an aquifer due to artificial recharge.

14. MULTIPLE WELL SYSTEMS

In a well field, when cone of depression of one well overlaps with the cone of depression of other wells, then the actual drawdown will be more than the drawdown calculated for the individual well (Fig. 13.2). In this case, the actual drawdown can be calculated using the principle of superposition of linear system [9].

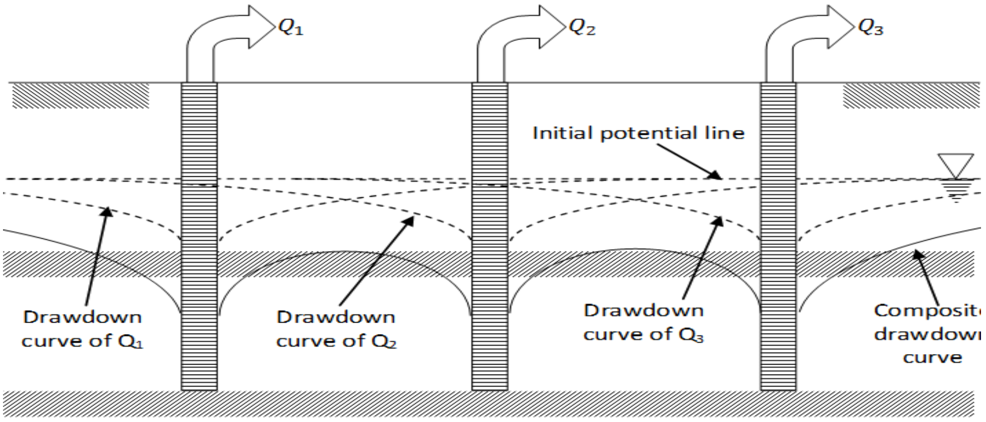

Fig. 14.1 Multiple well system

For a well field of *n* wells, the actual drawdown can be calculated as

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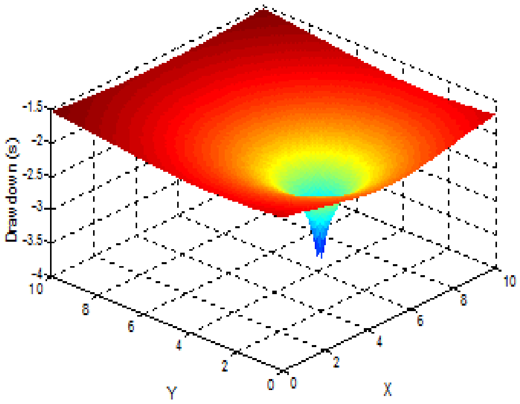
$$s_a(r,t) = s_1(r,t) + s_2(r,t) + s_3(r,t) + s_4(r,t) + s_5(r,t) + \cdots + s_n(r,t) \qquad (14.1)$$

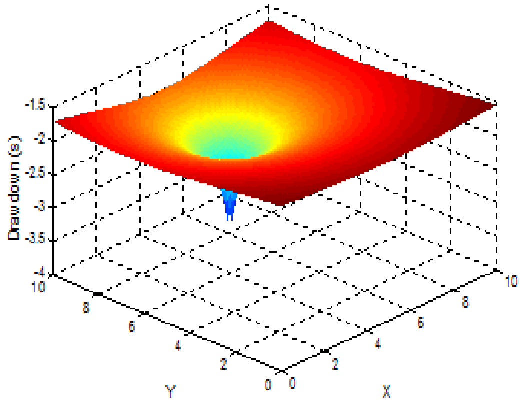
or, $$s_a(r,t) = \sum_{i=1}^{n} s_i(r,t)$$

Where S_a is the actual drawdown at a distance r at time t , S_i is the drawdown at that point caused by the discharge of the well i at time t, n is the number of wells in the well fields.

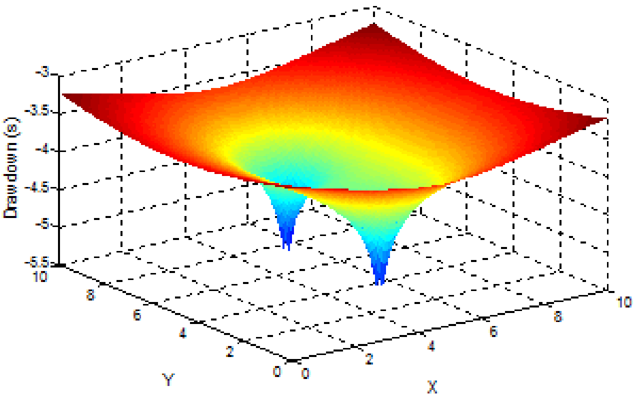
Fig. 14.2 explains the interference of cone of depression of two pumping wells. The coordinates of the two wells are (3,5) and (7,5). The individual cone of depression of the two wells are shown on Fig. 14.2 (a) and (b). The combine effect of the two wells can be obtained by adding the individual drawdown of the two wells, *i.e.* if drawdown of the first well is S_1 and the second well is S_2, the combine drawdown will be $S = S_1 + S_2$. The combine effect is shown in Fig. 14.2(c).



a) Drawdown of first well



(b) Drawdown of second well



(c) Combine drawdown

Fig. 14.2 Cone of depression of multiple wells system

15. WELLS NEAR AQUIFER BOUNDARIES

The assumption of infinite horizontal extend is no longer valid when water is pumped from a well near the aquifer boundary. Method of superposition can be used to implement the effect of aquifer boundary by adding a well at different location. The well that creates the same effect as boundary is called image well.

16. WELL NEAR A STREAM

Fig. 16.1 shows a well near a stream. In this case, the actual drawdown at the stream boundary will be zero as stream is considered as an infinite source. In order to maintain zero drawdown, an imaginary recharge well is considered at a distance equal to the distance between the pumping well and the stream boundary [8].

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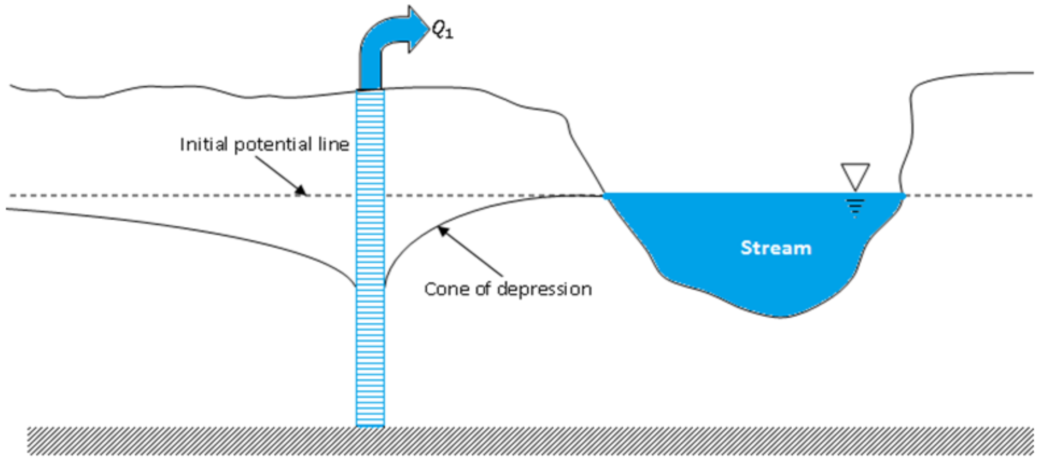

Fig. 16.1 Well near a stream

Fig. 16.2 shows an equivalent hydraulic system in an aquifer of infinite areal extend. For the equivalent hydraulic system, the time drawdown relationship for the pumping well and also for the imagery recharge well can be obtained separately. The actual drawdown can be obtained using the principle of superposition.

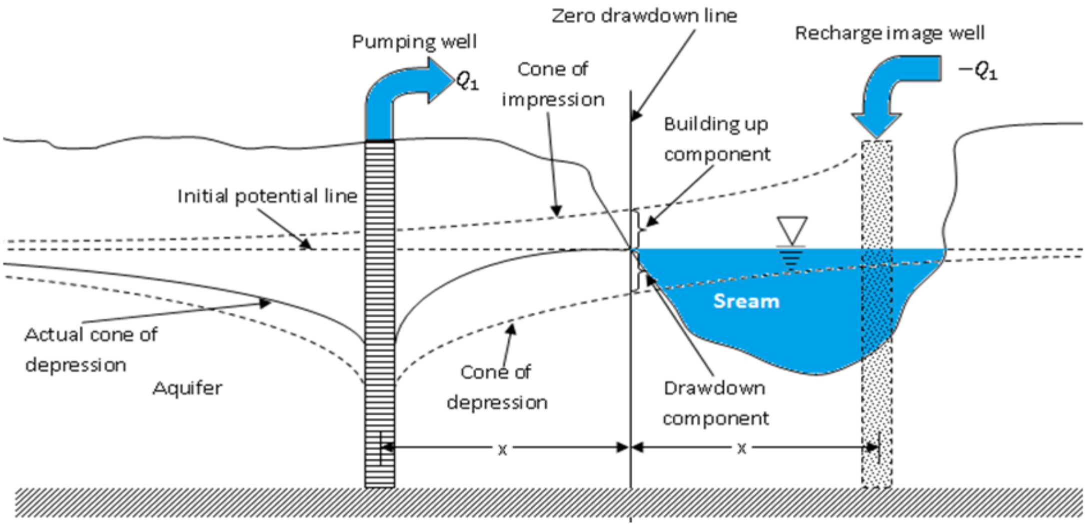

Fig. 16.2 Equivalent hydraulic system in a aquifer of infinite areal extend

Consider the Fig. 16.3 below. The pumping well is at a distance of x from the stream boundary. In order to calculate the actual drawdown at the observation location, an image well is

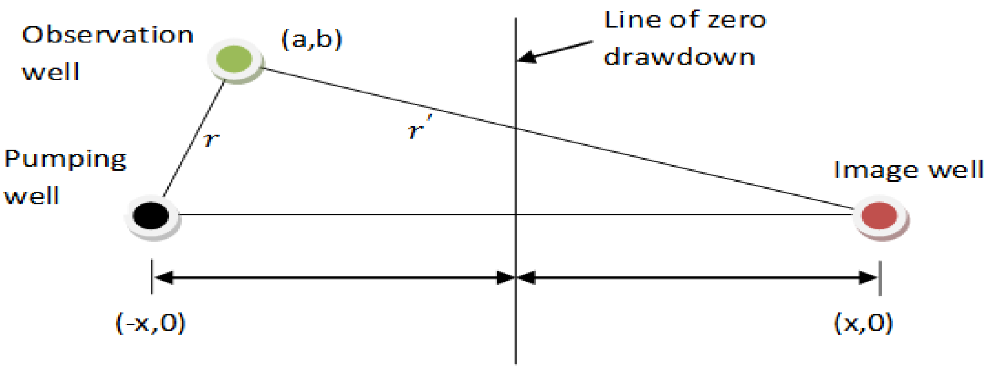

Fig. 16.3 Pumping well, Observation well and Image well

considered at a distance of x on the other side of the line of zero drawdown. The distance of the observation well from the pumping well is r and from the image well is r'.

For the steady state condition of a confined aquifer, the drawdown at the observation well can be obtained as

$$s(a,b) = \frac{Q}{2\pi T} ln\left(\frac{R}{r}\right) + \frac{-Q}{2\pi T} ln\left(\frac{R}{r'}\right) \quad (16.1)$$

$$\Rightarrow s(a,b) = \frac{Q}{2\pi T} ln\left(\frac{r'}{r}\right) \quad (16.2)$$

$$\Rightarrow s(a,b) = \frac{Q}{4\pi T} ln\left(\frac{(a+x)^2+b^2}{(a-x)^2+b^2}\right) \quad (16.3)$$

For the unsteady condition, the drawdown at r at any time t can be obtained as

$$s(r,t) = \frac{Q}{4\pi T} W\left(\frac{r^2 S_S}{4Tt}\right) + \frac{-Q}{4\pi T} W\left(\frac{r'^2 S_S}{4Tt}\right) \quad (16.4)$$

$$s(r,t) = \frac{Q}{4\pi T}\left[W\left(\frac{r^2 S_S}{4Tt}\right) - W\left(\frac{r'^2 S_S}{4Tt}\right)\right] \quad (16.5)$$

17. WELL NEAR AN IMPERMEABLE BOUNDARY

Fig. 17.1 shows a well near an impermeable boundary. In this case, the actual drawdown at the

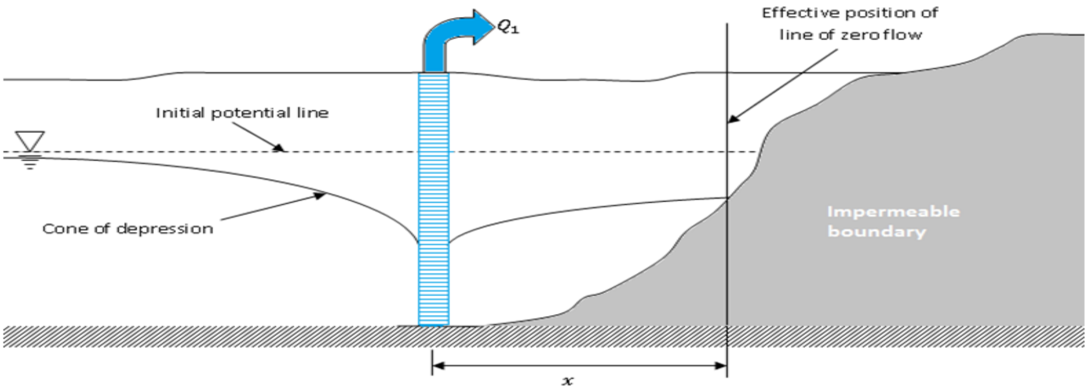

Fig. 17.1 Well near an impermeable boundary

impermeable boundary will be more than the drawdown calculated considering infinite areal extends of the aquifer medium. This problem can be solved by considering an imaginary pumping well at a distance equal to the distance between the pumping well and the image pumping well. Fig. 17.2 has shown the equivalent hydraulic system in an aquifer with infinite areal extent. For the equivalent hydraulic system, the time drawdown relationship for the pumping well and also for the imagery recharge well can be obtained separately. The actual drawdown can be obtained using the principle of superposition [8].

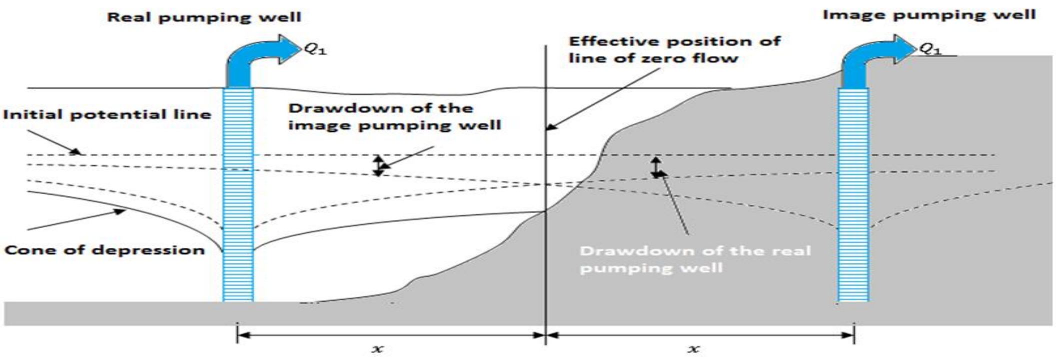

Fig. 17.2 Equivalent hydraulic system in a aquifer of infinite areal extend

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Consider the Fig. 17.3 below. The pumping well is at a distance of *x* from the impermeable boundary. In order to calculate the actual drawdown at the observation location, an image well is considered at a distance of *x* on the other side of the line of zero flow. The distance of the observation well from the pumping well is r and from the image well is r'. For the unsteady condition, the drawdown at a distance r at any time *t* can be obtained as,

$$s(r,t) = \frac{Q}{4\pi T} W\left(\frac{r^2 S_S}{4Tt}\right) + \frac{Q}{4\pi T} W\left(\frac{r'^2 S_S}{4Tt}\right) \quad (17.1)$$

$$s(r,t) = \frac{Q}{4\pi T}\left[W\left(\frac{r^2 S_S}{4Tt}\right) + W\left(\frac{r'^2 S_S}{4Tt}\right)\right] \quad (17.2)$$

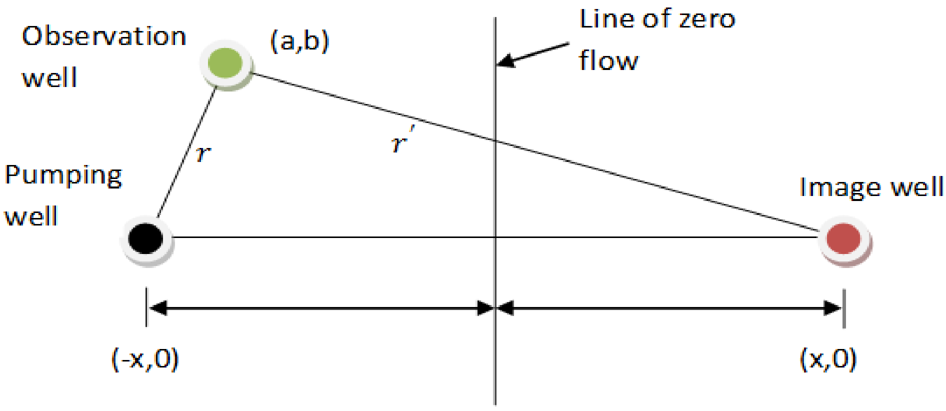

Fig. 17.3 Pumping well, observation well and image well

The transmissivity *(T)* of a confined aquifer can be calculated using the equation (8.14). This equation was derived for steady state condition. It may be noted that it is difficult to obtain steady state pumping drawdown data as one has to continue the pumping for longer period. The unsteady flow data can be used to calculate both hydraulic conductivity and transmissivity and storage coefficient of an aquifer. In this lecture we will mainly discuss the estimation of aquifer parameters using unsteady flow data.

18. TWO-DIMENSIONAL CONFINED AQUIFER

Large open cuts are a feature of many engineering projects. These are particularly prominent in open-cut mining, such as strip mining of coal. An important aspect of such operations is the analysis of slope stability for which an estimate of the groundwater discharge from the seams is required (Nguyen & Ngeyen, 1982). An excessive discharge will also make an efficient mining operation difficult [6].

The literature on the analysis of groundwater flow and methods of field testing is extensive but most of the solutions available were derived for well flow problems, e.g. Boulton (1954, 1965), de Wiest (1963), Glover (1966), Raudkivi & Callander (1976) , Sternberg (1969), Walton (1970). Mansur & Kaufman (1962) derived an "equivalent well" expression for an excavated pit and then applied well flow formulae. More recently the finite element method has become popular as a tool for solving groundwater flow problems, e.g. Neuman & Witherspoon (1970, 1971) , Wilson & Hamilton (1978) , Cushman et *al.* (1979). However, it is highly desirable to have a set of simple, though approximate, formulae for quick and easy estimation of groundwater flow into the above-mentioned large excavations.

The following is a presentation of some analytical solutions to the two-dimensional equations which describe the transient groundwater flow into large excavations. For confined aquifer flows the solutions are obtained by the Laplace transformation of the differential equations involved. The unconfined flow case is presented using the Dupuit approximation as originally given by Polubarinova-Kochina (1962).

The equation for unsteady Darcian flow of water in a confined elastic aquifer is

$$\nabla^2 h = \frac{S}{T}\frac{\partial h}{\partial t} \quad (18.1)$$

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where $\nabla^2 h = \partial^2 h/\partial x^2 + \partial^2 h/\partial y^2 + \partial^2 h/\partial z^2$; is piezometric head; $S = S_s b$ is the dimensionless storage coefficient; $S_s$is the specific storage; b is the thickness of the aquifer; $T = Kb$ is the transmissivity $m^2 s^{-1}$is the permeability (ms^{-1}) of the aquifer. Equation (24.1) can be written for the two-dimensional case as

$$\frac{\partial^2 s}{\partial x^2} = \frac{1}{\omega}\frac{\partial s}{\partial t} \qquad (18.2)$$

where $s = H_0 - h$ is the drawdown from the initial piezometric head H_0 and $\omega = T/S$.

Constant Drawdown

The boundary conditions are as follows.

Initially: $s(x,0) = 0$ (18.3)

at excavation face : $x = 0;\ s(0,t) = s_w$ (18.4)

and as $x \to \infty$: $s(\infty,t) = 0$ (18.5)

The Laplace transforms of equations (18.2)-(18.5) are

$$\frac{\partial^2 \bar{s}}{\partial x^2} = \frac{p}{\omega}\bar{s} \qquad (18.6)$$

$$\bar{s}(o,p) = \frac{s_w}{p} \qquad (18.7)$$

$$\bar{s}(\infty,t) = 0 \qquad (18.8)$$

giving the solution

$$\bar{s} = (s_w/p)\exp[-(p/\omega)^{\frac{1}{2}}x] \qquad (18.9)$$

The inverse Laplace transform of equation (18.9), the drawdown equation for this case, is

$$s = s_w \operatorname{erf}c[x/2(\omega t)^{\frac{1}{2}}] \qquad (18.10)$$

where erf c( ) is the complementary error function, defined as

$$\operatorname{erf}c(y) = 1 - \operatorname{erf}(y) = 1 - (2/\sqrt{\pi})\int_0^y e^{-v^2}\,dv$$

which is tabulated in mathematical handbooks (e.g. Abramovitz & Stegun, 1972) .

The discharge per unit length of the excavation face is

$$q = -T\left.\frac{\partial s}{\partial x}\right|_{x\to 0} \qquad (18.11)$$

Constant Discharge

The boundary conditions are as follows:

$$s(x,0) = 0 \qquad (18.12)$$

$$s(\infty,t) = 0 \qquad (18.13)$$

and the flow rate $(m^2 s^{-1})$ per unit length of the excavation face (one face)

$$q = -T\left.\frac{\partial s}{\partial x}\right|_{x\to 0} \qquad (18.14)$$

The Laplace-transform solution of equation (18.2) with these boundary conditions (equations (18.12)-(18.14)) is

$$\bar{s} = \frac{q}{t}\frac{\exp[-(p/\omega)^{\frac{1}{2}}x]}{p(p/\omega)^{\frac{1}{2}}} \qquad (18.15)$$

of which the inverse transform is

$$s = \frac{q\sqrt{\omega}}{T}\left\{2(t/\pi)^{\frac{1}{2}}\exp[-x^2/4\omega t] - \frac{x}{\sqrt{\omega}}\operatorname{erf} c\,\frac{x}{2(\omega t)^{\frac{1}{2}}}\right\} \qquad (18.16)$$

The total flow into the excavation from an aquifer initially under static conditions (horizontal piezometric surface) is

$$q = 2T\left.\frac{\partial s}{\partial x}\right|_{x\to 0} \qquad (18.17)$$

and from an aquifer with initial steady flow with gradient I

$$q = TI - T\left.\frac{\partial s}{\partial x}\right|_{x\to 0} \qquad (18.18)$$

19. LEAKY AQUIFERS

Leaky (semi-confined) aquifers are those where the individual aquifers (layers) are separated from each other by layers with very much lower permeability than the aquifer layers [6].

The unsteady flow equation, corresponding to equation (18.1), can be written as [9]

$$\nabla^2 s - \frac{s}{B^2} = \frac{S}{T}\frac{\partial s}{\partial t} \qquad (19.1)$$

where $s = H - h$ is the drawdown; $B = \sqrt{Kbb_1/K_1}$ is the leakage factor; K and b and K_1 and $b_1$are the permeability and thickness of the aquifer and aquitard, respectively.

Constant Drawdown

For two-dimensional flow equation (19.1) becomes

$$\frac{\partial^2 s}{\partial x^2} - \frac{s}{B^2} = \frac{1}{\omega}\frac{\partial s}{\partial t} \qquad (19.2)$$

With boundary conditions given by equations (18.3)-(18.5) the Laplace transform solution is

$$\bar{s} = (s_w/p)exp\,\{-[(p/\omega) + 1/B^2]^{\frac{1}{2}}x\} \qquad (19.3)$$

of which the inverse, the drawdown, is

$$s = s_w\exp(-x/B) - (\sqrt{\pi}/4)s_w[\exp(-x/B).\operatorname{erf} c(\xi) - \exp(x/B).\operatorname{erfc}(\eta)] \qquad (19.4)$$

where

$$\xi = \frac{(\omega t)^{\frac{1}{2}}}{B} - \frac{x}{2(\omega t)^{\frac{1}{2}}}$$

and

$$\eta = \frac{(\omega t)^{\frac{1}{2}}}{B} + \frac{x}{2(\omega t)^{\frac{1}{2}}}$$

At steady state, as t tends to infinity

$$\operatorname{erf} c\,(\xi) = \operatorname{erf} c(\eta) = 0$$

and

$$s = s_w\exp(-x/B) \qquad (19.5)$$

which is the solution of the steady state version of equation (19.2) i.e. where the right-hand side is equal to zero.

For large values of B equation (19.4) can be approximated to

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$$s = s_w\{\exp(-x/B) - (\sqrt{\pi}/4)erf\ c[x/2(\omega t)^{\frac{1}{2}}]\} \qquad (19.6)$$

Constant Discharge

Here the boundary conditions given by (18.12) and (18.13) apply to equation (19.2) and lead to the Laplace transform solution

$$\bar{s} = q/[Tp(p/\omega + 1/B^2)^{\frac{1}{2}}]exp - [(p/\omega + 1/B^2)^{\frac{1}{2}}]x \qquad (19.7)$$

of which the inverse is

$$s = (qB/T)\ exp\ (-x/B) - (\sqrt{\pi}/4)(qB/T)[\ exp\ (-x/B)\ \mathrm{erf}\, c\ (\xi) + \exp(x/B)\ \mathrm{erf}\, c(\eta) \qquad (19.8)$$

where the individual terms are as defined before.

20. CONCLUSION

Simple analytical solutions have been obtained for two-dimensional confined, leaky and unconfined groundwater flows. The confined and leaky aquifer solutions obtained by the Laplace transformation of the unsteady flow equation are in terms of the error function and its complementary part. The input parameters required for the solution are the piezometric head, the transmissivity and storage coefficient of the aquifer. The solutions provide a simple method for computing the phreatic surface at any time and from its slope the flow rate. The flow rate over a time interval is the difference of the areal integrals of the water table curves divided by the time interval.

REFERENCES

[1] Charles R. Fitts, Groundwater Science, Academic Press, An Imprint of Elsevier, 84 Theobalds Road, Landon WCIX 8RR, UK, 2002

[2] H. S. Carslaw and J. C. Jaeger, Conduction of Heat in Solids, Second Edition, Oxford University Press, Ely House, Landon 1959.

[3] Jacob Bear, Hydraulics of Groundwater, McGraw-Hill Series in Water Resources and Environmental Engineering, New York, 1979.

[4] J Crank, Mathematical Diffusion, Clarendon Press Oxford 1975.

[5] M. Necati ozisic, Heat Conduction, Second Edition, A Wiley-Interscience Publication, John Wiley & Sons, Inc. New York 1993.

[6] V. U. Nguyen and A. J. Raudkivi, transient two-dimensional groundwater flow, hydrological sciences-journal-des sciences hydrologiques, 28, 2, 6/1983.

[7] Cobb P.M., Mcelwee C.D., and Butt M.A. (1982), "Analysis of leaky aquifer pumping test data: An automated numerical solution using sensitivity analysis", Ground Water , 20(3), 325-333.

[8] Copper H.H. Jr., and Jacob C.E., (1946), “A generalized graphical method for evaluating formation constants and summarizing well field history”, Transaction of American Geophysical Union , 27, 526-534.

[9] Hantush M.S., (1956), "Analysis of data from pumping tests in leaky aquifers", Transaction of American Geophysical Union , 37, 702-714.

[10] Theis C.V. (1935), “The relation between the lowering of the piezometric surface and the rate and duration of discharge of a well using groundwater storage”, Transaction American Geophysical Union , 16, 519-524.

[11] Todd and Mays, (2011), "Groundwater hydrology", Wiley India Edition, Third edition, pp 195-196.